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Symmetric and Non-Symmetric Bases of Quantum Integrable Particle Systems with Long-Range Interactions

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We study in an algebraic manner the symmetric basis of the Calogero model and the non-symmetric basis of the corresponding Calogero model with distinguishable particles. The Rodrigues formulas are presented for the polynomial parts of both bases. The square norm of the non-symmetric basis is evaluated. Symmetrization of the non-symmetric basis reproduces the symmetric basis and enables us to calculate its square norm.

KEY WORDS: Quantum integrable system; long-range interaction; Calogero model; Hi–Jack polynomial; multivariable Hermite polynomial; Weyl group; Dunkl–Cherednik operator; Rodrigues formula.

1. INTRODUCTION

We call a quantum Hamiltonian system to be integrable when it has the same number of independent and mutually commutative conserved operators as its degrees of freedom. This definition is an extension of the Liouville theorem in classical mechanics. From a viewpoint of the range of interactions, we can classify integrable systems into two types, that is, systems with short-range interactions and those with long-range interactions.

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In 1971, Calogero introduced the quantum many-body system with inverse-square and harmonic interactions in a one-dimensional space.[3] He obtained the exact energy eigenvalue of the model through the separation of variables. Since then, there have been many works on the properties and the extensions of classical and quantum long-range interaction systems, including those by Sutherland, Moser, Olshanetsky and Perelomov.[22, 24, 25] We adopt the Calogero–Moser–Sutherland (CMS) models as the generic name for one-dimensional many-body systems with inverse-square long-range interactions. Their classical and quantum integrability has been studied extensively from various points of view. It is, however, rather recent that the eigenstates for some of quantum CMS models are investigated in a systematic manner. The aim of the present paper is a report on such a development.

In this article, we consider the quantum Calogero model described by the following Hamiltonian:

$$\hat{H}_{\mathrm{C}} := \frac{1}{2} \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2 \right) + \sum_{1 \leqslant j < k \leqslant N} \frac{g}{(x_j - x_k)^2} \tag{1.1}$$

We write the coupling parameter as $g = a(a-1)$ and assume $a \in \mathbb{R}_{\geqslant 1}$ and $\omega \in \mathbb{R}_{>0}$. Due to the symmetries of their variables $\{x_j\}$, the model is sometimes called the A_{N-1}-type Calogero model. Although we mostly deal with the Calogero model in what follows, the approach we shall take for this model can be extended to some of the other CMS models.[17–21, 33]

This paper is organized as follows. In Section 2, we introduce the A_{N-1}-type root system, the associated Weyl group and notations. These preliminaries imply extensions of the method to other quantum CMS models. In Section 3, we construct the symmetric basis of the Calogero model. The polynomial parts of the eigenstates are algebraically constructed. We call the obtained expressions the Rodrigues formulas. In Section 4, we present the non-symmetric basis of the Calogero model with distinguishable particles. The Rodrigues formula for the polynomial parts is derived. The square norm of the non-symmetric basis is evaluated also in an algebraic manner. We further discuss the relationship between the symmetric basis and the non-symmetric one. The final section is devoted to the summary.

2. PRELIMINARIES

In order to investigate the symmetric and non-symmetric bases of the Calogero model, we begin with a root system and the associated Weyl group. Let $\check{I} = \{1, 2, \ldots, N-1\}$ and $I = \{1, 2, \ldots, N\}$ be sets of indices and V

be an N-dimensional real vector space with positive definite bilinear form $\langle \cdot, \cdot \rangle$. We take an orthogonal basis $\{\varepsilon_j \mid j \in I\}$ of V such that $\langle \varepsilon_j, \varepsilon_k \rangle = \delta_{jk}$. We introduce the A_{N-1}-type root system $R := \{\varepsilon_j - \varepsilon_k \mid j, k \in I, j \neq k\} \subset V$ associated with the simple Lie algebra of type A_{N-1}. A root basis of R is defined by $\Pi := \{\alpha_j = \varepsilon_j - \varepsilon_{j+1} \mid j \in \check{I}\}$. R_+ denotes the set of positive roots relative to Π and $R_- := -R_+$.

We consider a reflection on V with respect to the hyperplane orthogonal to a root $\alpha \in R$, and express it by $s_\alpha(\mu) := \mu - \langle \alpha^\vee, \mu \rangle \alpha$ where $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$ is a coroot corresponding to $\alpha \in R$. The reflections $\{s_j := s_{\alpha_j} \mid \alpha_j \in \Pi\}$ generate the symmetric group $\mathfrak{S}_N$ which is isomorphic to the A_{N-1}-type Weyl group $W \simeq \mathfrak{S}_N$. For each $w \in W$, we define the following set of positive roots:

$$R_w := R_+ \cap w^{-1} R_-$$

If we take a shortest expression $w = s_{j_l} \cdots s_{j_2} s_{j_1}$, the set R_w is expressed by

$$R_w = \{\alpha_{j_1}, s_{j_1}(\alpha_{j_2}), \ldots, s_{j_1} s_{j_2} \cdots s_{j_{l-1}}(\alpha_{j_l})\}$$

We introduce lattices $P := \bigoplus_{j \in I} \mathbb{Z}_{\geqslant 0}\, \varepsilon_j$ and $P_+ := \{\mu = \sum_{j \in I} \mu_j \varepsilon_j \in P \mid \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_N \geqslant 0\}$ whose elements are called a composition and a partition respectively. Let $W(\mu) := \{w(\mu), w \in W\}$ be the W-orbit of $\mu \in P$. In a W-orbit $W(\mu)$, there exists a unique partition $\mu^+ \in P_+$ such that $\mu = w(\mu^+) \in P (w \in W)$. As usual, the Weyl vector is defined by

$$\rho := \tfrac{1}{2} \sum_{\alpha \in R_+} \alpha = \tfrac{1}{2} \sum_{j \in I} (N - 2j + 1)\, \varepsilon_j$$

We identify the elements of the lattice P with those of polynomial ring with N variables over $\mathbb{C}$, $x^\mu := x_1^{\mu_1} x_2^{\mu_2} \cdots x_N^{\mu_N} \in \mathbb{C}[x]$. Then, in terms of the coordinate exchange operators $\{K_{jk}\}$ defined by

$$(K_{jk} f)(\ldots, x_j, \ldots, x_k, \ldots) := f(\ldots, x_k, \ldots, x_j, \ldots)$$

the action of the operators $\{K_{jj+1} \mid j \in \check{I}\}$ on $\mathbb{C}[x]$ are written as

$$K_{jj+1}(x^\mu) = x^{s_j(\mu)}, \qquad \text{for} \quad x^\mu \in \mathbb{C}[x]$$

3. SYMMETRIC BASIS OF CALOGERO MODEL

The energy spectrum of the Calogero model (1.1) is given by[25]

$$E_\mu = \omega\, |\mu| + \tfrac{1}{2} \omega N (Na + (1 - a))$$

where $|\mu| := \sum_{j\in I}\mu_j$ is the weight of the partition $\mu = \sum_{j\in I}\mu_j\varepsilon_j \in P_+$. One readily sees that there exists a large degeneracy in the spectrum. The degeneracy implies the necessity of additional conserved operators. The quantum Lax formulation provides the commutative conserved operators for the Calogero model.[26, 27] We found that the second conserved operator in fact solves the large degeneracy; we obtain the symmetric orthogonal basis of the Calogero model by diagonalizing the first and second conserved operators.[29]

We start from the following commutative operators called the Dunkl operators,[7]

$$\begin{aligned} \nabla_j &:= \frac{\partial}{\partial x_j} + a \sum_{k(\neq j)} \frac{1}{x_j - x_k}(1 - K_{jk}), \qquad && \text{for} \quad j \in I \\ [\nabla_j, \nabla_k] &= 0, && \text{for} \quad j, k \in I \end{aligned} \tag{3.1}$$

The commutative operators of Dunkl type play a crucial role for the quantum integrability of the CMS models. We call such approach the exchange operator formulation[23] or the Dunkl–Cherednik operator formulation.[4] It was shown that the commutativity of these operators comes from the Yang–Baxter relations.[12, 13] For the CMS models with interactions of trigonometric type, the Yang–Baxter relations appear in the form of the affine Hecke algebras.[4, 5, 17, 21]

For the non-interaction case, the Calogero Hamiltonian (1.1) reduces to that of the quantum harmonic oscillators. In analogous way to the definition of the creation, annihilation and number operators of the quantum harmonic oscillators, we introduce the creation-, annihilation- and number-like operators of the Calogero model,

$$\alpha_j^* := x_j - \frac{1}{2\omega}\nabla_j, \qquad \alpha_j := \nabla_j, \qquad n_j := \alpha_j^*\alpha_j, \qquad \text{for} \quad j \in I \tag{3.2}$$

They satisfy

$$\begin{aligned} &[\alpha_j, \alpha_k] = [\alpha_j^*, \alpha_k^*] = 0, && \text{for} \quad j, k \in I \\ &[\alpha_j, \alpha_k^*] = \delta_{jk}\left(1 + a \sum_{l \neq j} K_{jl}\right) - a(1 - \delta_{jk}) K_{jk}, \qquad && \text{for} \quad j, k \in I \\ &[n_j, n_k] = a(n_k - n_j) K_{jk}, && \text{for} \quad j, k \in I \end{aligned} \tag{3.3}$$

We define the inner product:

$$\langle f, g \rangle_{a,\omega} := \int_{-\infty}^{\infty} \prod_{j\in I} dx_j \, |\phi_{\mathrm{g}}(x)|^2 f(x)\, g(x), \qquad \text{for} \quad f, g \in \mathbb{C}[x] \tag{3.4}$$

where ϕ_{g} is the ground state wave function,

$$\phi_{\mathrm{g}}(x) := \prod_{1 \leqslant j < k \leqslant N} |x_j - x_k|^a \exp\left(-\tfrac{1}{2}\omega \sum_{j \in I} x_j^2\right) \tag{3.5}$$

Here we have taken the symmetric (bosonic) ground state wave function. Note that the creation-like operators $\{\alpha_j^*\}$ and the annihilation-like operators $\{\alpha_j\}$ are in the relation: $\langle \alpha_j f, g \rangle_{a,\omega} = \langle f, \alpha_j^* g \rangle_{a,\omega}$. Then the number-like operators $\{n_j\}$ are Hermitian with respect to the inner product (3.4), that is, $\langle n_j f, g \rangle_{a,\omega} = \langle f, n_j g \rangle_{a,\omega}$. Due to a nontrivial relation between the creation- and annihilation-like operators, all the symmetric states constructed by the creation-like operators are not orthogonal with respect to the inner product (3.4). To make an orthogonal basis, we use the commutative conserved operators which are Hermitian with respect to the inner product (3.4). The power sums of the number-like operators $\{n_j\}$ give a commutative family of operators $\{I_l \mid l \in I\}$,

$$\begin{aligned} I_l &:= \sum_{j \in I} (n_j)^l \Big|_{\mathrm{Sym}}, \qquad && \text{for} \quad l \in I \\ [I_l, I_m] &= 0, && \text{for} \quad l, m \in I \end{aligned} \tag{3.6}$$

where $|_{\mathrm{Sym}}$ means that their operands are restricted to the symmetric function space. Through the similarity transformation by use of ϕ_{g}, we find that the operators

$$\hat{I}_l := \phi_{\mathrm{g}} \circ I_l \circ \phi_{\mathrm{g}}^{-1}, \qquad \text{for} \quad l \in I \tag{3.7}$$

are the commutative conserved operators of the Calogero model. It is to be remarked that they are equivalent to those derived by the quantum Lax formulation.[26, 27] The first conserved operator indeed corresponds to the Calogero Hamiltonian $\omega \hat{I}_1 = \hat{H}_{\mathrm{C}}$. There exists a family of symmetric polynomials $J_\mu \in \mathbb{C}[x]^W$, $(\mu \in P_+)$ which are joint eigenvectors of operators I_1 and I_2,

$$\begin{aligned} J_\mu &= m_\mu + \sum_{\nu <^{\mathrm{d}} \mu \text{ or } |\nu| < |\mu|} v_{\mu\nu}\left(a, \frac{1}{2\omega}\right) m_\nu, \qquad \text{for} \quad \mu \in P_+ \\ I_1 J_\mu &= \sum_{j \in I} \mu_j J_\mu \\ I_2 J_\mu &= \sum_{j \in I} (\mu_j^2 + a(N+1-2j)\,\mu_j)\, J_\mu \end{aligned} \tag{3.8}$$

which are named Hi–Jack polynomials.[30] In the defining relations (3.8), we have used the monomial symmetric functions $m_\mu(x) := \sum_{\nu \in W(\mu)} x^\nu \in \mathbb{C}[x]^W$, $(\mu \in P_+)$ and the dominance order $<^{\mathrm{d}}$ on P_+,

$$\nu \leqslant^{\mathrm{d}} \mu \quad (\nu, \mu \in P_+) \Leftrightarrow |\nu| = |\mu|, \quad \text{and} \quad \sum_{k=1}^{l} \nu_k \leqslant \sum_{k=1}^{l} \mu_k \qquad \text{for all} \quad l \in I$$

Thus the first and second conserved operators of the Calogero model $\hat{I}_1$ and $\hat{I}_2$ have the joint eigenvectors Φ_μ, $(\mu \in P_+)$ which can be expressed by products of the ground state wave function ϕ_{g} and the Hi–Jack polynomials J_μ,

$$\Phi_\mu(x) = \phi_{\mathrm{g}}(x)\, J_\mu(x)$$

In fact, the states Φ_μ are the joint eigenstates of all the conserved operators $\{\hat{I}_l\}$.[31] It is interesting to observe that the Hi–Jack polynomial reduces to the Jack polynomial[9, 16] in the limit, $\omega \to \infty$.

Lapointe and Vinet showed the Rodrigues formula for the Jack polynomial by means of the raising operators.[14] Extending their method to the Calogero model, we present the Rodrigues formula for the Hi–Jack polynomial.[30, 31] We introduce the raising operators $\{B_k^+ \mid k \in I\}$ as

$$B_k^+ := \sum_{J \subseteq I,\, |J| = k} \alpha_J^* n_{k,J}, \qquad \text{for} \quad k \in \check{I}, \qquad B_N^+ := \alpha_1^* \alpha_2^* \cdots \alpha_N^* \tag{3.9}$$

where

$$\alpha_J^* := \prod_{j \in J} \alpha_j^*, \qquad\qquad \text{for} \quad J \subseteq I$$

$$n_{m,J} := (n_{j_1} + ma)(n_{j_2} + (m+1)\,a) \cdots (n_{j_k} + (m+k-1)\,a), \qquad \text{for} \quad k = |J|$$

and $|J|$ means the number of elements in a subset $J \subseteq I$. Applying the raising operators $\{B_k^+\}$ to the reference state $J_0 = 1$, we can show that the Rodrigues formula for the Hi–Jack polynomial J_μ, $(\mu \in P_+)$ is given by

$$\begin{aligned} J_\mu &= C_\mu^{-1} (B_N^+)^{\mu_N} (B_{N-1}^+)^{\mu_{N-1} - \mu_N} \cdots (B_1^+)^{\mu_1 - \mu_2} J_0 \\ C_\mu &:= \prod_{k \in I} (a)_{\mu_k - \mu_{k+1}} (2a + \mu_{k-1} - \mu_k)_{\mu_k - \mu_{k+1}} (ka + \mu_1 - \mu_k)_{\mu_k - \mu_{k+1}} \end{aligned} \tag{3.10}$$

where C_μ is the coefficient of the top term m_μ, $(\beta)_n = \beta(\beta+1)\cdots(\beta+n-1)$ and $(\beta)_0 = 1$. By use of the Rodrigues formula (3.10), we proved that the expansion coefficients are polynomials of a and $1/2\omega$ with integer coefficients (integrality).[30, 31] The Hi–Jack polynomial is found to be a multivariable generalization of the Hermite polynomial, which was introduced by Lassalle and Macdonald from the viewpoint of a deformation of an orthogonal polynomial.[15]

4. NONSYMMETRIC BASIS OF CALOGERO MODEL WITH DISTINGUISHABLE PARTICLES

Motivated by the Haldane-Shastry spin chains, the Calogero models with spin degrees of freedom were introduced.[8, 28] The Hamiltonian of the spin Calogero model is given by

$$\hat{H}_{\mathrm{Spin}} := \frac{1}{2} \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2 \right) + \sum_{1 \leqslant j < k \leqslant N} \frac{1}{(x_j - x_k)^2} a(a - P_{jk}) \tag{4.1}$$

where the operator P_{jk} is the spin exchange operator. In the case of SU(2) spin-1/2, for example, $P_{jk} = (1 + \sigma_j \cdot \sigma_k)/2$ with the Pauli spin matrices $\sigma = (\sigma^x, \sigma^y, \sigma^z)$. The eigenfunction of the spin Calogero model is expressed by a linear combination of products of the orbital and spin parts. When the whole eigenfunction is symmetric under the exchange of particles, i.e., $K_{jk} P_{jk} = 1$, the spin Calogero model is mapped to

$$\hat{\mathscr{H}}_{\mathrm{C}} := \frac{1}{2} \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2 \right) + \sum_{1 \leqslant j < k \leqslant N} \frac{1}{(x_j - x_k)^2} a(a - K_{jk}) \tag{4.2}$$

Recall that K_{jk} is the coordinate exchange operator. It is important to notice that K_{jk} is identified with P_{jk} only in the symmetric case. We refer to the model (4.2) as the Calogero model with distinguishable particles and construct the eigenstates which correspond to the orbital parts of the spin Calogero model (4.1).

The Dunkl–Cherednik operator formulation provides the commutative conserved operators for the Calogero model with distinguishable particles (4.2) as well. By use of the operators (3.2), we introduce the differential operators $\{d_j \mid j \in I\}$,

$$d_j := \alpha_j^* \alpha_j + a \sum_{k=j+1}^{N} K_{jk} \tag{4.3}$$

which we call the Cherednik operators. The Cherednik operators $\{d_j\}$ and the coordinate exchange operators $\{K_{jj+1}\}$ satisfy the following relations:

$$\begin{aligned} [d_j, d_k] &= 0, && \text{for} \quad j, k \in I \\ d_j K_{j,\,j+1} - K_{j,\,j+1} d_{j+1} &= a, && \text{for} \quad j, k \in \check{I} \\ d_j K_{k,\,k+1} &= K_{k,\,k+1} d_j, && \text{for} \quad j \neq k, k+1 \end{aligned} \tag{4.4}$$

Note that they are commutative and are not W-invariant. For the later discussions, we write

$$d^\lambda := \sum_{j \in I} \lambda_j d_j, \qquad \text{for} \quad \lambda \in P \tag{4.5}$$

The power sums of the Cherednik operators $\{d_j\}$ (without any restriction to their operands) give a family of independent commutative operators,

$$\mathscr{I}_l := \sum_{j \in I} (d_j)^l, \qquad \text{for} \quad l \in I \tag{4.6}$$

The first conserved operator $\mathscr{I}_1$ corresponds to

$$\begin{aligned} \mathscr{H}_{\mathrm{C}} &:= \omega \sum_{j \in I} \left(d_j - \frac{1}{2}(N-1)\,a \right) \\ &= \sum_{j=1}^{N} \left(-\frac{1}{2} \frac{\partial^2}{\partial x_j^2} + \omega x_j \frac{\partial}{\partial x_j} \right) \\ &\quad - a \sum_{1 \leqslant j < k \leqslant N} \left(\frac{1}{x_j - x_k} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) + \frac{K_{jk} - 1}{(x_j - x_k)^2} \right) \end{aligned} \tag{4.7}$$

which changes into the original Hamiltonian (4.2) by use of ϕ_{g},

$$\mathscr{H}_{\mathrm{C}} = \omega \sum_{j=1}^{N} \phi_{\mathrm{g}} \circ \left(d_j - \frac{1}{2}(N-1)\,a \right) \circ \phi_{\mathrm{g}}^{-1}$$

Since the Cherednik operators $\{d_j\}$ commute with the transformed Hamiltonian $\mathscr{H}_{\mathrm{C}}$, we consider the operators $\{\phi_{\mathrm{g}} \circ d_j \circ \phi_{\mathrm{g}}^{-1}\}$ as the commutative conserved operators of the Calogero model with distinguishable particles. We note that $\{d_j\}$ are Hermitian with respect to the inner product (3.4).

The joint eigenvectors for the commutative Cherednik operators $\{d_j\}$ are the non-symmetric multivariable Hermite polynomials $j_\mu \in \mathbb{C}[x]$, $(\mu \in P)$,[1]

$$\begin{aligned} j_\mu(x) &= x^\mu + \sum_{\nu \preccurlyeq \mu \text{ or } |\nu| < |\mu|} v_{\mu\nu}\left(a, \frac{1}{2\omega}\right) x^\nu \\ d^\lambda j_\mu &= (\langle \lambda, \mu + a\rho(\mu)\rangle + a(N-1)/2)\, j_\mu \end{aligned} \tag{4.8}$$

where the triangularity is defined by the order $\preccurlyeq$ on P:

$$\nu \preccurlyeq \mu \qquad (\nu, \mu \in P) \Leftrightarrow \begin{cases} \text{(i)} & \nu^+ \neq \mu^+ \text{ and } \nu^+ <^{\mathrm{d}} \mu^+ \\ \text{(ii)} & \nu^+ = \mu^+ \text{ and the first} \\ & \text{non-vanishing difference} \\ & \mu_j - \nu_j > 0 \end{cases} \tag{4.9}$$

We denote by w_μ the shortest element of W such that $w_\mu^{-1}(\mu) \in P_+$ to define $\rho(\mu) := w_\mu(\rho)$. All the eigenspaces of the Cherednik operators $\{d_j\}$ are one-dimensional in the sense that the eigenvalues of $\{d_j\}$ are non-degenerate. One sees that the polynomials (4.8) are not symmetric under the exchange of variables $\{x_j\}$. Applying the coordinate exchange operators $\{K_{jj+1} \mid j \in I\}$ to the non-symmetric Hermite polynomials j_μ, $(\mu \in P)$, we find that

$$K_{jj+1} j_\mu = \begin{cases} \dfrac{a}{\langle \alpha_j^\vee, \mu + a\rho(\mu)\rangle} j_\mu + j_{s_j(\mu)}, & \text{if } \langle \alpha_j^\vee, \mu\rangle < 0 \\ j_\mu, \quad \text{if } \langle \alpha_j^\vee, \mu\rangle = 0 \\ \dfrac{a}{\langle \alpha_j^\vee, \mu + a\rho(\mu)\rangle} j_\mu + \dfrac{\langle \alpha_j^\vee, \mu + a\rho(\mu)\rangle^2 - a^2}{\langle \alpha_j^\vee, \mu + a\rho(\mu)\rangle^2} j_{s_j(\mu)}, \\ \qquad \text{if } \langle \alpha_j^\vee, \mu\rangle > 0 \end{cases}$$

Since $\{d_j\}$ are Hermitian operators, the polynomials j_μ are proved to be orthogonal with respect to the inner product (3.4), i.e., $\langle j_\mu, j_\nu\rangle_{a,\omega} = \delta_{\mu\nu} \|j_\mu\|^2$. In fact, the nonsymmetric multivariable Hermite polynomials form an orthogonal basis in $\mathbb{C}[x]$.

We present the Rodrigues formula which produces the non-symmetric Hermite polynomials.[32, 33] We introduce the Knop–Sahi operators $\{e, e^*\}$[11] and the braid operators $\{S_j \mid j \in \check{I}\}$ defined by

$$\begin{aligned} e := \alpha_1 K_{1,2} K_{2,3} \cdots K_{N-1,N}, \qquad & e^* = K_{N-1,N} \cdots K_{2,3} K_{1,2} \alpha_1^* \\ S_j := [K_{jj+1}, d_j] & \end{aligned} \tag{4.10}$$

The operators $\{e, e^*\}$ were first introduced by Baker and Forrester.[2] Useful relations among those operators are

$$\begin{aligned}
&S_j S_{j+1} S_j = S_{j+1} S_j S_{j+1}, \qquad \text{for} \quad 1 \leqslant j \leqslant N-2 \\
&S_j e^* = e^* S_{j+1}, \qquad \text{for} \quad 1 \leqslant j \leqslant N-2, \quad S_{N-1}(e^*)^2 = (e^*)^2 S_1 \\
&S_j^2 = a^2 - (d_j - d_{j+1})^2, \qquad \text{for} \quad j \in \check{I} \\
&S_j^* = -S_j, \qquad \text{for} \quad j \in \check{I}, \quad e^* e = d_N
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&S_j d_j = d_{j+1} S_j, \qquad \text{for} \quad j \in \check{I} \\
&d_j e^* = e^* d_{j+1}, \qquad \text{for} \quad j \in \check{I}, \qquad d_N e^* = e^*(d_1 + 1)
\end{aligned} \tag{4.12}$$

The first relation in (4.11) is called braid relation.[34] The relations (4.12) indicate that the operators $\{S_j, e^*\}$ intertwine the eigenspaces of $\{d_j\}$. We define the raising operators $\{A_\mu^* \mid \mu \in P_+\}$ by

$$\begin{aligned}
&A_\mu^* := (A_1^*)^{\mu_1 - \mu_2} (A_2^*)^{\mu_2 - \mu_3} \cdots (A_N^*)^{\mu_N} \\
&A_j^* := (S_j S_{j+1} \cdots S_{N-1} e^*)^j \qquad \text{for} \quad j \in I
\end{aligned} \tag{4.13}$$

From the relations which are obtained from (4.11) and (4.12),

$$\begin{aligned}
&d_j A_k^* = \begin{cases} A_k^*(d_j + 1), & \text{if} \quad 1 \leqslant j \leqslant k \\ A_k^* d_j, & \text{if} \quad k < j \leqslant N \end{cases} \\
&[A_j^*, A_k^*] = 0, \qquad \text{for} \quad j, k \in I
\end{aligned} \tag{4.14}$$

we see that $A_\mu^* j_\nu$, $(\mu, \nu \in P_+)$ coincides with $j_{\mu+\nu}$ up to a constant factor. Hence we obtain the joint eigenvectors of $\{d_j\}$,

$$\begin{aligned}
&\tilde{j}_\mu := A_\mu^* j_0 = c_\mu j_\mu, \qquad \text{for} \quad \mu \in P_+ \\
&d_j \tilde{j}_\mu = (\mu_j + a(N-j))\, \tilde{j}_\mu = (\langle \varepsilon_j, \mu + a\rho \rangle + a(N-1)/2)\, \tilde{j}_\mu
\end{aligned} \tag{4.15}$$

where $j_0 = 1$ is the reference state. Since all the eigenspaces of $\{d_j\}$ are one-dimensional, we can identify $\tilde{j}_\mu$, $(\mu \in P_+)$ with the non-symmetric multivariable Hermite polynomials with a partition $\mu \in P_+$ up to a constant factor. The coefficients of the top term of $\tilde{j}_\mu$ in (4.15) are directly calculated as

$$c_\mu := \prod_{\alpha \in R_+} \prod_{l=1}^{\langle \alpha^\vee, \mu \rangle} (l + a\langle \alpha^\vee, \rho \rangle) \tag{4.16}$$

To construct the non-symmetric multivariable Hermite polynomials with a composition $\mu \in P$, we apply the braid operators to the eigenvector j_μ with a partition $\mu^+ \in P_+$ lying in $W(\mu)$,

$$j_\mu = c_{w_\mu}^{-1} S_{w_\mu} j_{\mu^+}, \qquad c_{w_\mu} := \prod_{\alpha \in R_{w_\mu}} \frac{\langle \alpha^\vee, \mu^+ + a\rho \rangle^2 - a^2}{\langle \alpha^\vee, \mu^+ + a\rho \rangle} \tag{4.17}$$

where $S_{w_\mu} := S_{j_l} \cdots S_{j_2} S_{j_1}$ is determined by a shortest expression $w = s_{i_l} \cdots s_{i_2} s_{i_1}$. We thus obtain the Rodrigues formula for the non-symmetric multivariable Hermite polynomials with a generic composition $\mu = w_\mu(\mu^+)$ $(\in P)$,

$$j_\mu = c_{w_\mu}^{-1} c_\mu^{-1} S_{w_\mu} A_{\mu^+}^* j_0 \tag{4.18}$$

Then the nonsymmetric orthogonal basis ϕ_μ, $(\mu \in P)$ of the Calogero model with distinguishable particles is given by

$$\phi_\mu(x) = \phi_{\mathrm{g}}(x)\, j_\mu(x) = \phi_{\mathrm{g}}(x)\, c_{w_\mu}^{-1} c_\mu^{-1} S_{w_\mu} A_{\mu^+}^* j_0(x) \tag{4.19}$$

We evaluate the square norm of the non-symmetric basis ϕ_μ, $(\mu \in P)$,

$$\|\phi_\mu\|^2 := \int_{-\infty}^{\infty} \prod_{j \in I} dx_j\, |\phi_\mu(x)|^2 = \langle j_\mu, j_\mu \rangle_{a,\omega}$$

To this end, we consider square norm of the non-symmetric multivariable Hermite polynomial j_μ with respect to the inner product (3.4) through its Rodrigues formula. First, we calculate the square norm of j_μ with a partition $\mu \in P_+$. The adjoint operators of the raising operators $\{A_\mu^*\}$ with respect to the inner product (3.4) are given by

$$\begin{aligned} A_\mu &:= A_N^{\mu_N} \cdots A_2^{\mu_2 - \mu_3} A_1^{\mu_1 - \mu_2}, && \text{for} \quad \mu \in P_+ \\ A_j &= (eS_{N-1}^* \cdots S_{j+1}^* S_j^*)^j, && \text{for} \quad j \in I \end{aligned} \tag{4.20}$$

Using the square norm of the reference state,

$$\langle 1, 1 \rangle_{a,\omega} = \frac{(2\pi)^{N/2}}{(2\omega)^{1/2\, N(Na + (1-a))}} \prod_{j \in I} \frac{\Gamma(1 + ja)}{\Gamma(1 + a)} \tag{4.21}$$

which is a limiting case of the Selberg integral, we obtain

$$\begin{aligned}
&\langle j_\mu, j_\mu\rangle_{a,\omega}\\
&\quad = c_\mu^{-2}\langle A_\mu^* j_0, A_\mu^* j_0\rangle_{a,\omega}\\
&\quad = c_\mu^{-2}\langle j_0, A_\mu A_\mu^* j_0\rangle_{a,\omega}\\
&\quad = \frac{(2\pi)^{N/2}}{(2\omega)^{1/2\,N(Na+(1-a))}} \prod_{j\in I} \Gamma(\mu_j+1+a(N-j))\\
&\qquad \times\left(\prod_{k=j+1}^{N} \frac{\Gamma(\mu_j-\mu_k+1+a(k-j+1))\,\Gamma(\mu_j-\mu_k+1+a(k-j-1))}{\Gamma(\mu_j-\mu_k+1+a(k-j))^2}\right)\\
&\quad = \frac{(2\pi)^{N/2}}{(2\omega)^{1/2\,N(Na+(1-a))}} \prod_{j\in I} \Gamma(\langle \varepsilon_j, \mu+a\rho\rangle+1+a(N-1)/2)\\
&\qquad \times \prod_{\alpha\in R_+} \frac{\Gamma(\langle \alpha^\vee, \mu+a\rho\rangle+a+1)\,\Gamma(\langle \alpha^\vee, \mu+a\rho\rangle-a+1)}{\Gamma(\langle \alpha^\vee, \mu+a\rho\rangle+1)^2}
\end{aligned} \tag{4.22}$$

for $\mu\in P_+$. Next, we calculate the square norm of j_μ with a general composition $\mu\in P$. The quadratic relations of $\{S_j\}$ in (4.11) and the formula (4.17) lead to the following relations:

$$\frac{\langle j_\mu, j_\mu\rangle_{a,\omega}}{\langle j_{\mu^+}, j_{\mu^+}\rangle_{a,\omega}} = \prod_{\alpha\in R_{w_\mu}} \frac{\langle \alpha^\vee, \mu^+ + a\rho\rangle^2}{\langle \alpha^\vee, \mu^+ + a\rho\rangle^2 - a^2} \tag{4.23}$$

Equation (4.23) with (4.22) gives the square norm of j_μ with $\mu\in P$.

Now we discuss the relationship between the symmetric basis constructed in the previous section and the non-symmetric one. One sees that the non-symmetric multivariable Hermite polynomial with a composition lying in the same W-orbit has the same eigenvalue of the similarity transformed Hamiltonian (4.7),

$$\mathscr{H}_{\mathrm{C}}\, j_\mu = \omega\,|\mu^+|\, j_\mu, \qquad \text{for}\quad \mu\in W(\mu^+),\ \mu^+\in P_+ \tag{4.24}$$

We take the following linear combination of j_μ, $(\mu\in W(\mu^+), \mu^+\in P_+)$:

$$\begin{aligned}
J_{\mu^+} &= \sum_{\mu\in W(\mu^+)} b_{\mu^+\mu}\, j_\mu\\
b_{\mu^+\mu} &= \prod_{\alpha\in R_{w\mu}} \frac{\langle \alpha^\vee, \mu^+ + a\rho\rangle - a}{\langle \alpha^\vee, \mu^+ + a\rho\rangle}
\end{aligned} \tag{4.25}$$

whose coefficients $b_{\mu^+\mu}$ are determined by the conditions $K_{jj+1} J_{\mu^+} = J_{\mu^+}$ for all $j\in\check{I}$. The polynomials J_μ, $(\mu\in P_+)$ are nothing else but the multivariable Hermite (Hi–Jack) polynomials, as is seen from $J_{\mu^+}\in\mathbb{C}[x]^W$ and their orthogonality with respect to the inner product (3.4). Thus we recover the symmetric basis Φ_μ, $(\mu \in P_+)$ of the Calogero model (1.1),

$$\Phi_\mu(x) = \phi_{\mathrm{g}}(x) \sum_{\mu \in W(\mu^+)} b_{\mu^+\mu} c_w^{-1} c_\mu^{-1} S_{w_\mu} A^*_{\mu^+} j_0(x) \tag{4.26}$$

We evaluate the square norm of the symmetric basis ϕ_μ, $(\mu \in P_+)$,

$$\|\Phi_\mu\|^2 := \int_{-\infty}^{\infty} \prod_{j \in I} dx_j \, |\Phi_\mu(x)|^2 = \langle J_\mu, J_\mu \rangle_{a,\omega}$$

To do this, we show a new proof of the inner product identity for the multivariable Hermite polynomials J_μ through symmetrization of the square norms of the non-symmetric multivariable Hermite polynomials j_μ. In relating the square norm of the non-symmetric basis to that of the symmetric basis, the following identity is useful:

$$\sum_{\mu \in W(\mu^+)} \prod_{\alpha \in R_{w_\mu}} \frac{\langle \alpha^\vee, \mu^+ + a\rho \rangle - a}{\langle \alpha^\vee, \mu^+ + a\rho \rangle + a} = \prod_{\alpha \in R_+} \frac{\langle \alpha^\vee, \mu^+ + a\rho \rangle}{\langle \alpha^\vee, \mu^+ + a\rho \rangle + a} \tag{4.27}$$

We proved this identity (4.27) in our previous paper[17] to establish symmetrization of the nonsymmetric Macdonald polynomials. By use of the orthogonality of j_μ, the square norm $\langle j_\mu, j_\mu \rangle_{a,\omega}$ (4.22), (4.23) and the relation (4.25), we obtain

$$\begin{aligned}
&\langle J_{\mu^+}, J_{\mu^+} \rangle_{a,\omega} \\
&\quad = \sum_{\mu \in W(\mu^+)} (b_{\mu^+\mu})^2 \langle j_\mu, j_\mu \rangle_{a,\omega} \\
&\quad = \sum_{\mu \in W(\mu^+)} (b_{\mu^+\mu})^2 \frac{\langle j_\mu, j_\mu \rangle_{a,\omega}}{\langle j_{\mu^+}, j_{\mu^+} \rangle_{a,\omega}} \langle j_{\mu^+}, j_{\mu^+} \rangle_{a,\omega} \\
&\quad = \sum_{\mu \in W(\mu^+)} \prod_{\alpha \in R_{w_\mu}} \frac{\langle \alpha^\vee, \mu^+ + a\rho \rangle - a}{\langle \alpha^\vee, \mu^+ + a\rho \rangle + a} \langle j_{\mu^+}, j_{\mu^+} \rangle_{a,\omega} \\
&\quad = \prod_{\alpha \in R_+} \frac{\langle \alpha^\vee, \mu^+ + a\rho \rangle}{\langle \alpha^\vee, \mu^+ + a\rho \rangle + a} \langle j_{\mu^+}, j_{\mu^+} \rangle_{a,\omega} \\
&\quad = \frac{(2\pi)^{N/2}}{(2\omega)^{1/2\, N(Na + (1-a))}} \prod_{j \in I} \Gamma(\mu_j^+ + 1 + a(N-j)) \\
&\qquad \times \prod_{1 \leqslant j < k \leqslant N} \frac{\Gamma(\mu_j^+ - \mu_k^+ + a(k-j+1))\, \Gamma(\mu_j^+ - \mu_k^+ + a(k-j-1)+1)}{\Gamma(\mu_j^+ - \mu_k^+ + a(k-j))\, \Gamma(\mu^+{}_j - \mu_k^+ + a(k-j)+1)}
\end{aligned} \tag{4.28}$$

The square norms of the multivariable Hermite polynomials have been also obtained by other methods.[6, 10] Our symmetrization method can be applied in a systematic manner to other polynomials.[17, 21]

5. SUMMARY

We summarize the results. First, we have studied the symmetric orthogonal basis of the Calogero model. The commutative conserved operators of the Calogero model are obtained through the Dunkl–Cherednik operator formulation. The multivariable Hermite (Hi–Jack) polynomials are defined as the joint eigenvectors of the commutative operators I_1 and I_2. We have presented the Rodrigues formula for the multivariable Hermite polynomials. The explicit formula enables us to prove that the expansion coefficients are polynomials of a and $1/2\omega$ with integer coefficients (integrality). Second, we have studied the non-symmetric orthogonal basis of the Calogero model with distinguishable particles and have clarified the relationship with the symmetric basis. The Rodrigues formula also exists for the non-symmetric multivariable Hermite polynomials from which the square norm of the non-symmetric basis is evaluated. Symmetric linear combinations of the non-symmetric multivariable Hermite polynomials recover the multivariable Hermite polynomials. We have presented a new proof of the square norms of the symmetric basis through symmetrization of the non-symmetric counterparts.

To conclude this paper, two comments are in order:

(i) Although some of the results are known, a consistent presentation is new.

(ii) As a typical example of quantum integrable particle systems with long-range interactions, we have mostly discussed the Calogero model (1.1). However, the method is general and is applicable to other polynomials.[17–21, 33]

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